7.2　偏导数与全微分

7.2 偏导数(partial derivative)与全微分

7.2.1 偏导数的概念

1 偏导数的定义

(1) $f(x,y)$在点$P_0(x_0,y_0)$处的偏导数

定义 7.2.1 设函数$z=f(x,y)$在点(x_0,y_0)的某一邻域内有定义，当y固定在y_0而x在x_0处有增量Δx时，相应地函数有增量

$$f(x_0+\Delta x,y_0)-f(x_0,y_0),$$

如果$\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ (1) 存在，

则称此极限为函数 $z = f(x, y)$ 在点 (x_0, y_0) 处对 x 的偏导数，记为

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}},\quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,\ y_0)},\quad \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}},\quad \left.z_x\right|_{\substack{x=x_0\\ y=y_0}} \text{ 或 } f_x(x_0, y_0)$$

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

例 1 求 $z=x^2+3xy+y^2$ 在点(1,2)处对 x 的偏导数.

解 $$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}}=\lim_{\Delta x\to 0}\frac{(1+\Delta x)^2+3(1+\Delta x)\cdot 2+2^2-11}{\Delta x}$$

$$=\left.\frac{df(x,2)}{dx}\right|_{x=1}\qquad f(x,2)=x^2+3x\cdot 2+4$$

$$f_x(x_0,y_0)=\left.\frac{df(x,y_0)}{dx}\right|_{x=x_0}$$

同理可定义函数$z = f(x,y)$在点(x_0,y_0)处对y的偏导数，为

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

记为$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$，$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$，$\left.z_y\right|_{\substack{x=x_0\\y=y_0}}$或$f_y(x_0,y_0)$.

即$f_y(x_0,y_0) = \lim\limits_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$

（2）偏导函数

如果函数$z=f(x,y)$在区域 D内任一点(x,y)处对x的偏导数都存在，那么这个偏导数就是x、y的函数，称为函数$z=f(x,y)$对自变量x的偏导数,记作$\frac{\partial z}{\partial x}$，$\frac{\partial f}{\partial x}$，$z_x$或$f_x(x,y)$.

$$f_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

类似定义函数$z=f(x,y)$对自变量y的偏导数，记作

$\frac{\partial z}{\partial y}$,$\frac{\partial f}{\partial y}$，$z_y$或$f_y(x,y)$.

$$f_y(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

注: $$f_x(x_0,y_0)=f_x(x,y)\Big|_{\substack{x=x_0\\ y=y_0}}$$

$$f_y(x_0,y_0)=f_y(x,y)\Big|_{\substack{x=x_0\\ y=y_0}}$$

(3) 偏导数概念可推广到二元以上的函数

如$u=f(x,y,z)$在(x,y,z)处

$$f_x(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x},$$

$$f_y(x,y,z)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y},$$

$$f_z(x,y,z)=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}.$$

2. 偏导数的计算

仍然是一元函数的求导公式和求导法则，对某一个自变量求偏导时，**把其余的自变量看作常量.**

例 1 求 $z = x^2 + 3xy + y^2$ 在点(1,2)处的偏导数.

解 $\dfrac{\partial z}{\partial x} = 2x + 3y \qquad \dfrac{\partial z}{\partial y} = 3x + 2y$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}} = 2\times1 + 3\times2 = 8 \qquad \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=2}} = 3\times1 + 2\times2 = 7$$

例 2 设$z = x^y (x > 0, x \neq 1)$，

求证 $\dfrac{x}{y}\dfrac{\partial z}{\partial x} + \dfrac{1}{\ln x}\dfrac{\partial z}{\partial y} = 2z.$

证明 $\dfrac{\partial z}{\partial x} = yx^{y-1}$, $\quad \dfrac{\partial z}{\partial y} = x^y \ln x$,

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = \frac{x}{y} yx^{y-1} + \frac{1}{\ln x} x^y \ln x$$

$$= x^y + x^y = 2z.$$

原结论成立.

例3. $u = x^{y^z} = x^{(y^z)}$ ，求u_x, u_y, u_z $3^{(3^2)} \neq (3^3)^2$

解： $u_x = y^z \cdot x^{y^z - 1} = \dfrac{1}{x} \cdot y^z \cdot x^{y^z}$；

$$u_y = x^{y^z} \cdot \ln x \cdot \frac{\partial}{\partial y}(y^z) = x^{y^z} \cdot \ln x \cdot z \cdot y^{z-1}$$

$$= \frac{z}{y} \ln x \cdot y^z \cdot x^{y^z}$$

或 对幂指函数两边取对数 $\ln u = y^z \ln x$

$$\frac{u_y}{u} = zy^{z-1} \ln x \Rightarrow u_y = \frac{z}{y} \ln x \cdot y^z \cdot x^{y^z}$$

$$u_z = \frac{\partial}{\partial z}(e^{y^z \ln x}) = x^{y^z} \cdot y^z \cdot \ln y \ln x$$

(2) 求$f_x(x_0, y_0)$时，可先将y_0代入得

$$f(x,y_0)=\varphi(x),\ \text{再求}\frac{d\varphi}{dx},\ \text{即}\frac{d\varphi}{dx}=\frac{df(x,y_0)}{dx},$$

最后再将x_0代入.

例5 $f(x,y)=x^2+(y-1)\arcsin\sqrt{\dfrac{x}{y}}$,

求$f_x(2,1)$.

解 $f(x,1)=x^2$, $\quad f_x(x,1)=\dfrac{df(x,1)}{dx}=2x$;

$f_x(2,1)=4$

(3)求分界点、不连续点处的偏导数只能用定义求

例 6 设 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

求 $f(x,y)$的偏导数.

解 当$(x,y)\neq(0,0)$时,

$$f_x(x,y)=\frac{y(x^2+y^2)-2x\cdot xy}{(x^2+y^2)^2}=\frac{y(y^2-x^2)}{(x^2+y^2)^2},$$

$$f_y(x,y)=\frac{x(x^2-y^2)}{(x^2+y^2)^2},$$

当$(x,y)=(0,0)$时， 按定义可知：

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0,$$

$$\because\ f(\Delta x,0)=\frac{\Delta x\cdot 0}{(\Delta x)^2+0^2}=0 \qquad f_y(0,0)=0$$

例 6 设 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

求 $f(x,y)$的偏导数.

例 6 设 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

求 $f(x,y)$的偏导数.

$$f_x(x,y)=\begin{cases}\dfrac{y(y^2-x^2)}{(x^2+y^2)^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases},$$

$$f_y(x,y)=\begin{cases}\dfrac{x(x^2-y^2)}{(x^2+y^2)^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}.$$

3 偏导数存在与连续的关系

一元函数中在某点可导 $\longrightarrow$ 连续，

多元函数中在某点偏导数存在 $\xrightarrow{?}$ 连续，

例如，函数 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$，

依定义知在(0,0)处，$f_x(0,0)=f_y(0,0)=0.$

但函数在该点处并不连续. 偏导数存在 $\not\longrightarrow$ 连续.

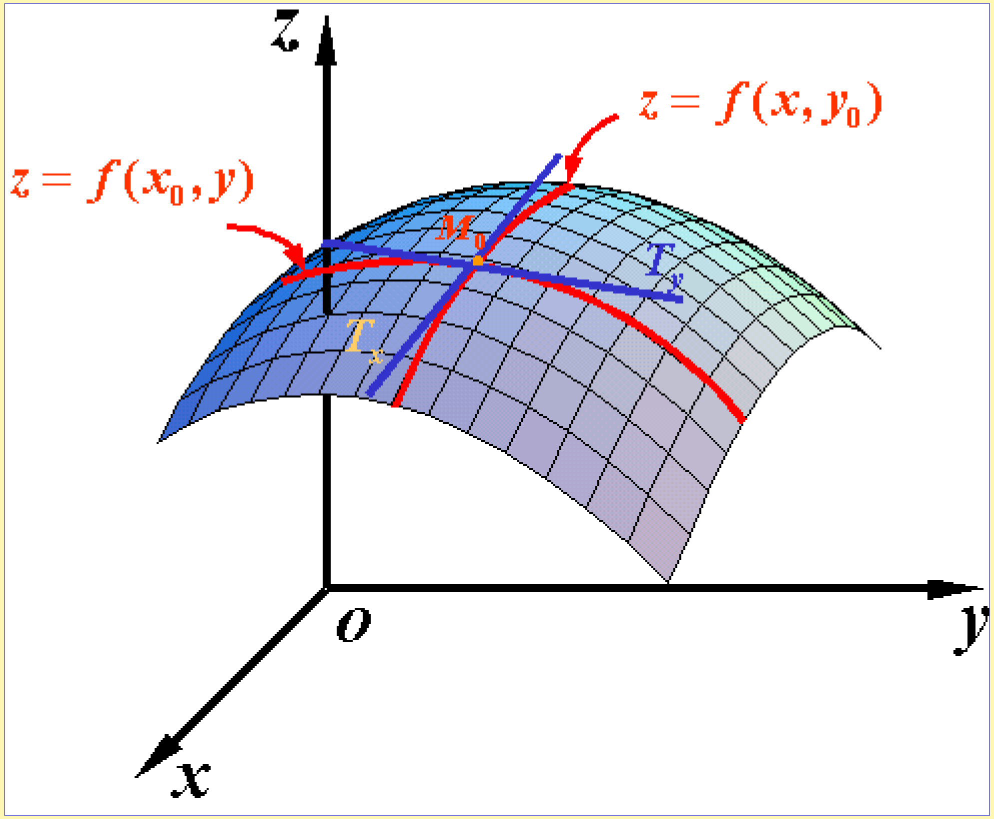
连续 $\not\longrightarrow$ 偏导数存在：如：$z=\sqrt{x^2+y^2}$ 在(0,0)处

7.2.2 偏导数的几何意义

设 $M_0(x_0, y_0, f(x_0, y_0))$ 为曲面 $z = f(x, y)$ 上一点，

$$f_x(x_0, y_0)$$

$$= \left.\frac{df(x, y_0)}{dx}\right|_{x=x_0}$$



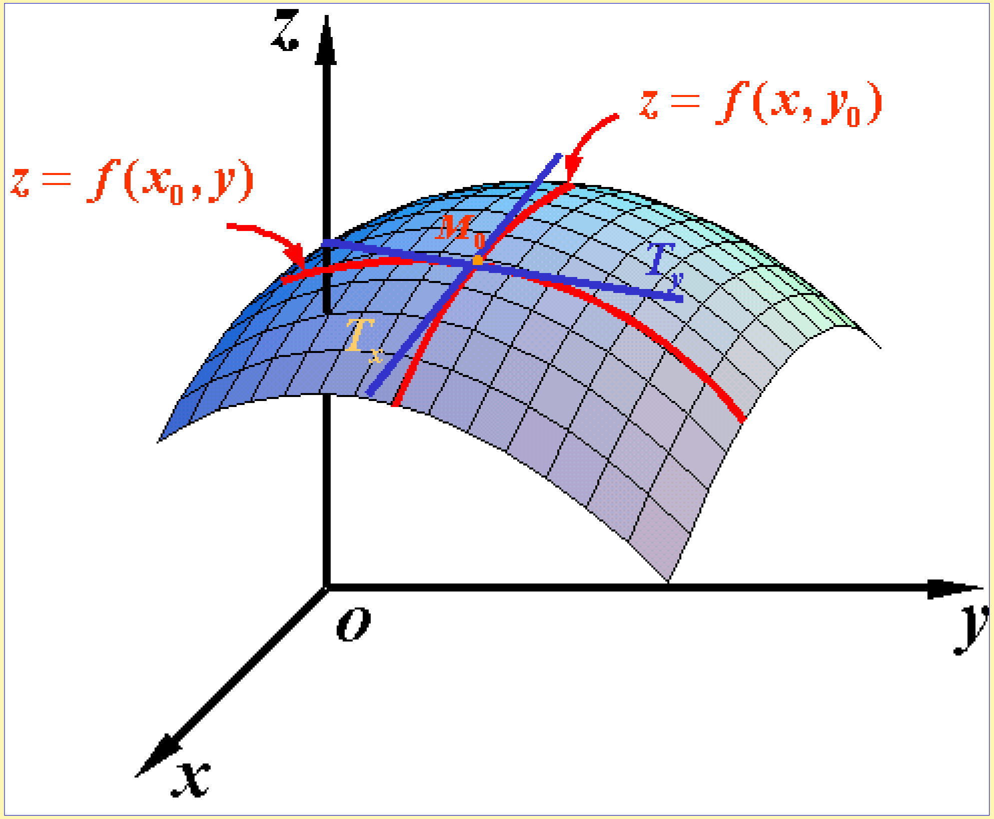
$f(x, y_0)$是曲面被平面$y = y_0$所截得的曲线的方程，

$f_x(x_0, y_0)$是该曲线在点M_0处的切线M_0T_x对x轴的斜率

几何意义:

$$f_y(x_0, y_0) = \left.\frac{df(x_0, y)}{dy}\right|_{y=y_0}$$

偏导数 $f_y(x_0, y_0)$就是曲面被平面$x = x_0$所截得的曲线在点M_0处的切线M_0T_y对y轴的斜率.



7.2.3 高阶偏导数　二阶及二阶以上的偏导数

函数$z=f(x,y)$，$\dfrac{\partial z}{\partial x}$仍是关于 x和 y 的函数

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

纯偏导

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y)$$

混合偏导

例7 设$z = x^3y^2 - 3xy^3 - xy + 1$，

求$\frac{\partial^2 z}{\partial x^2}$、$\frac{\partial^2 z}{\partial y \partial x}$、$\frac{\partial^2 z}{\partial x \partial y}$、$\frac{\partial^2 z}{\partial y^2}$及$\frac{\partial^3 z}{\partial x^3}$.

解 $\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$, $\frac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$;

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2,\quad \frac{\partial^3 z}{\partial x^3} = 6y^2,\quad \frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy;$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x^2y - 9y^2 - 1,$$

$$\frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 - 1.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}\ ?$$

问题： 混合偏导数都相等吗？

具备怎样的条件才能使混合偏导数相等？

定理 7.2.1 如果函数$z = f(x, y)$的两个二阶混合偏导数$\frac{\partial^2 z}{\partial y \partial x}$及$\frac{\partial^2 z}{\partial x \partial y}$在区域 D 内连续，那么在该区域内这两个二阶混合偏导数必相等.

例8 证明函数 $u=\frac{1}{r}$ 满足拉普拉斯方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$ 其中 $r=\sqrt{x^2+y^2+z^2}$

证明 $$u=\frac{1}{\sqrt{x^2+y^2+z^2}}=(x^2+y^2+z^2)^{-\frac{1}{2}},$$

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x=-x\cdot(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 u}{\partial x^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-x\cdot(-\frac{3}{2})(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x$$

$$=-\frac{1}{r^3}+\frac{3x^2}{r^5}.$$

例8 证明函数 $u=\dfrac{1}{r}$ 满足方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad \text{其中}\ r=\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{r^3}+\frac{3x^2}{r^5}$$

由于函数关于自变量的对称性，所以

因此 $\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{1}{r^3}+\dfrac{3y^2}{r^5}$, $\dfrac{\partial^2 u}{\partial z^2}=-\dfrac{1}{r^3}+\dfrac{3z^2}{r^5}$.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{r^3}+\frac{3(x^2+y^2+z^2)}{r^5}=-\frac{3}{r^3}+\frac{3r^2}{r^5}=0$$

7.2.4 全微分

1. 增量、全增量及偏微分

由一元函数微分学中增量与微分的关系得

$$\boxed{f(x+\Delta x,y_0)-f(x,y_0)} \approx \boxed{f_x(x,y_0)\Delta x}$$

$$\boxed{f(x,y+\Delta y)-f(x,y)} \approx \boxed{f_y(x,y)\Delta y}$$

二元函数 二元函数

对x和对y的偏增量 对x和对y的偏微分

$$\Delta z=f(x+\Delta x,\ y+\Delta y)-f(x,\ y) \qquad (1)$$

叫做函数在点$(x，y)$对应于自变量增量Δx、Δy的全增量.

2．全微分的定义

定义 7.2.2　如果函数$z=f(x,y)$在点(x,y)的全增量$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$可以表示为$\Delta z=A\Delta x+B\Delta y+o(\rho)$，其中$A,B$不依赖于$\Delta x,\Delta y$而仅与$x,y$有关，$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则称函数$z=f(x,y)$在点$(x,y)$可微分，$A\Delta x+B\Delta y$称为函数$z=f(x,y)$在点$(x,y)$的全微分，记为$dz$，即

$$dz=A\Delta x+B\Delta y.$$

函数若在某区域 D 内各点处处可微分，则称这函数在 D 内可微分.

函数$z=f(x,y)$在点(x,y)可微分, 则函数在该点连续.

可微: $\Delta z = A\Delta x + B\Delta y + o(\rho)$,

要证: $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f(x+\Delta x, y+\Delta y) = f(x,y)$

$$= \lim_{\rho\to 0}[f(x,y)+\Delta z] \Leftrightarrow \lim_{\rho\to 0}\Delta z = 0$$

3 可微的必要条件

定理 7.2.2（必要条件） 如果函$z=f(x,y)$在点(x,y)可微分，则该函数在点(x,y)的偏导数$\dfrac{\partial z}{\partial x}$、$\dfrac{\partial z}{\partial y}$必存在，且函数$z=f(x,y)$在点$(x,y)$的全微分$dz=\dfrac{\partial z}{\partial x}\Delta x+\dfrac{\partial z}{\partial y}\Delta y$.

$$dz = A\Delta x + B\Delta y = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

证明 如果函数$z = f(x, y)$在点$P(x, y)$可微分,

$P'(x+\Delta x, y+\Delta y) \in P$的某个邻域

$\Delta z = A\Delta x + B\Delta y + o(\rho)$ 总成立,

当$\Delta y = 0$时，上式仍成立，此时$\rho = |\Delta x|$,

$$f(x+\Delta x, y) - f(x, y) = A \cdot \Delta x + o(|\Delta x|),$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = A = \frac{\partial z}{\partial x},$$

同理可得 $B = \dfrac{\partial z}{\partial y}.$

4. 偏导存在不是函数可微的充分条件

一元函数可微等价于可导，

而多元函数偏导存在不能推出可微。

例如 $$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$$

$f(x,y)$在点原点处偏导存在，但 $f(x,y)$在点原点处不连续，所以$f(x,y)$在点原点处一定不可微。

5. 函数可微的充分条件

定理 7.2.3（充分条件） 如果函数 $z=f(x,y)$的偏导数$\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$在点$(x,y)$连续，则该函数在点$(x,y)$可微分.

证明 $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$$= \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + o(\rho)\ ?$$

$$\Delta z = f_x(x,y)\Delta x + \varepsilon_1 \Delta x + f_y(x,y)\Delta y + \varepsilon_2 \Delta y$$

当$\Delta x \to 0, \Delta y \to 0$时，$\varepsilon_1, \varepsilon_2 \to 0$

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + o(\rho) \approx dz$$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

取 $z = x$, 则 $\Delta x = dx$　　取 $z = y$, 则 $\Delta y = dy$

习惯上，记全微分为 $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$

全微分的定义可推广到三元及三元以上函数

$$u = u(x,\ y,\ z) \quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

6. 全微分的计算

方法：

（1）先求$f_x(x,y)$、$f_y(x,y)$，判断$f(x,y)$的可微性
(利用充分条件）

（2）$dz = f_x(x，y)dx + f_y(x，y)dy$

几类微分：(*i*) $P(x,y)$处的微分；

（*ii*）$P_0(x_0,y_0)$处的微分；

（*iii*）$P_0(x_0, y_0)$处且dx，dy给定时的微分

例9.（1）计算$z = x^2y+y^3$的全微分；

（2）计算$z = x^2y+y^3$在点(2，1)处的全微分；

（3）计算$z = x^2y+y^3$在点(2，1)处相应于$\Delta x=0.1$，$\Delta y=-0.1$时的全微分。

解 (1) $$\frac{\partial z}{\partial x} = 2xy, \frac{\partial z}{\partial y} = x^2 + 3y^2$$

$$dz = 2xydx + (x^2 + 3y^2)dy$$

(2) $$\left.\frac{\partial z}{\partial x}\right|_{(2,1)} = 4, \quad \left.\frac{\partial z}{\partial y}\right|_{(2,1)} = 7 \qquad dz = 4dx + 7dy$$

(3) $\Delta x=0.1$，$\Delta y=-0.1$， $\therefore dz = 0.4 - 0.7 = -0.3$

$$\Delta z = f(2+0.1, 1-0.1) - f(2,1) \approx dz$$

例 10 计算函数$u = x + \sin\frac{y}{2} + e^{yz}$的全微分.

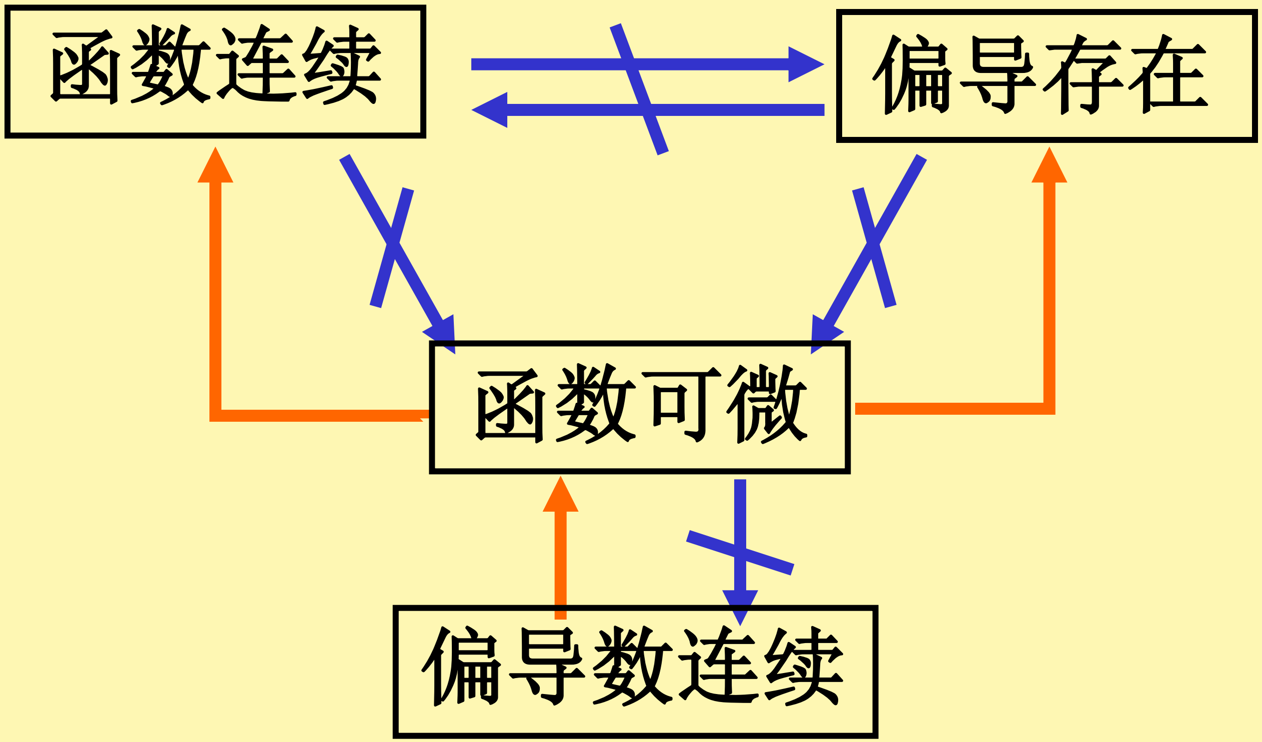
解 $\frac{\partial u}{\partial x} = 1,\quad \frac{\partial u}{\partial y} = \frac{1}{2}\cos\frac{y}{2} + ze^{yz},$

$\frac{\partial u}{\partial z} = ye^{yz},$

所求全微分

$$du = dx + \left(\frac{1}{2}\cos\frac{y}{2} + ze^{yz}\right)dy + ye^{yz}dz.$$

7　多元函数连续、可导、可微的关系



$$f(x,y)=\begin{cases}\left(x^2+y^2\right)\sin\left(\dfrac{1}{x^2+y^2}\right), & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

在原点可微，但偏导不连续

小结 本节主要讨论了多元函数的偏导数、高阶偏导数及全微分的概念。

注意：$\dfrac{\partial z}{\partial x}$是一个整体记号．$\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial y}\cdot\dfrac{\partial y}{\partial z}\neq 1$

本节要求理解多元函数的偏导数、高阶偏导数及全微分的概念；了解多元函数偏导数的几何意义；了解多元函数可微的充分与必要条件以及多元函数的连续、可导、可微的关系。熟练掌握多元函数的偏导数与全微分的计算。